# Inverse spectral theory and the Minkowski problem for the surface of revolution

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**I am going to talk about our joint work (in preparation)**

**Inverse spectral theory and the Minkowski problem for the surface of revolution**

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**The plan of the talk**

- **Spectral data mapping**
- **The surface of revolution**
- **Liouville transform — global properties**
- **Main results I. Spectral data mapping**
- **Main results II. Curvature mapping and the Minkowski problem**

# Section 1. Spectral data mapping

**Inverse spectral problem for $-(d/dx)^2 + q(x)$ on $(0,1)$**

**Borg (1946), Levinson (1949), Marchenko (1950), M.G.Krein (1951), Gel'fand-Levitan (1951), Marcenko-Ostrovski (1975),**

**Trubowitz-Isaacson (1983), Trubowitz-Issacson-Mckean (1984), Trubowitz-Dahlberg (1984), Trubowitz-Pöschel (1987)**

# Analitic functions in $\infty$-dimensions

Let $\mathrm{H}_1$, $\mathrm{H}_2$ be complex Hilbert spaces. A map

$$f : \mathrm{H}_1 \longrightarrow \mathrm{H}_2$$

is differentiable at $x \in \mathrm{H}_1$ if there exists a bounded linear operator $d_x f : \mathrm{H}_1 \longrightarrow \mathrm{H}_2$ such that

$$\|f(x+h) - f(x) - d_x f(h)\| = o(\|h\|).$$

**Gradient**. If $f : L^2(0,1) \longrightarrow \mathrm{C}$, $d_q f$ is a bounded linear functional on $L^2(0,1)$. So, by Riesz' theorem, there exists $g(x) \in L^2(0,1)$ such that

$$d_q f(v) = (v, g) = \int_0^1 v(x)\overline{g(x)}dx.$$

We denote

$$\overline{g(x)} = \frac{\partial f}{\partial q(x)}.$$

For example, if $\lambda_n(q)$ is the $n$-th eigenvalue of the operator $y \longrightarrow -y'' + q(x)y$ on $(0,1)$ with Dirichlet B. C., we can define

$$\frac{\partial \lambda_n}{\partial q(x)}.$$

**Definiton**. A function $f : \mathrm{H}_1 \longrightarrow \mathrm{H}_2$ is **analytic** if $f$ is continuously differentiable on $\mathrm{H}_1$.

For an analytic function, Cauchy's formula holds

$$f(x + zh) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{f(x+\zeta h)}{\zeta - z} d\zeta.$$

**Consider**

$$S_p y = -y''(x) + p(x)y(x), \quad 0 < x < 1,$$

**with Dirichlet B. C.**

$$y(0) = y(1) = 0.$$

**Suppose**

$$p \in \mathcal{H}_0 = \left\{ q(x) \in L^2_{\mathrm{R}}((0,1))\,;\, \int_0^1 q(x)dx = 0 \right\}.$$

**Then $S_p$ has a discrete spectrum**

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots,$$

$$\sigma_n = (\pi n)^2 + \widetilde{\sigma}_n, \quad \widetilde{\sigma} = (\widetilde{\sigma}_n)_{n=1}^{\infty} \in \ell^2,$$

**with associated eigenvectors $y_n(x)$. Letting**

$$\phi_n = \log \left| \frac{y_n'(1)}{y_n'(0)} \right|,$$

$$\mathcal{M}_1 = \Big\{ (h_n)_{n=1}^{\infty} \in \ell^2 \, ; \\ (\pi)^2 + h_1 < (2\pi)^2 + h_2 < \cdots < (n\pi)^2 h_n < \cdots \Big\},$$

$$\ell_1^2 = \left\{ (h_n)_{n=1}^{\infty} \in \ell^2 \, ; \sum_{n=1}^{\infty} n^2 |h_n|^2 < \infty \right\},$$

$$\mathcal{H}_0^{even} = \{ q \in \mathcal{H}_0 \, ; \, q(x) = q(1-x) \} \, ,$$

**we have the following theorem.**

## Theorem

**(1) The mapping**

$$\Phi : \mathcal{H}_0 \ni p \longrightarrow (\widetilde{\sigma}, \phi) \in \mathcal{M}_1 \times \ell_1^2$$

**is a real analytic onto isomorphism.**

**(2) For the even potential, the mapping**

$$\mathcal{H}_0 \ni p \longrightarrow \widetilde{\sigma} \in \mathcal{M}_1$$

**is a real analytic onto isomorphism.**

# Section 2. The surface of revolution

$\mathbf{R}^{m+2} = \mathbf{R}^1 \times \mathbf{R}^{m+1} \ni (x, y)$

The surface of revolution $M$ is defined by

$$y = f(x)\omega, \quad \omega \in S^m, \quad x \in I = [0, x_0].$$

Assume that $f(x) > 0$.

The induced metric on $M$ is

$$ds^2 = (1 + f'(x)^2)(dx)^2 + f(x)^2 g_{S^m}.$$

By the change of variable

$$\frac{dt}{dx} = \sqrt{1 + f'(x)^2},$$

we have

$$ds^2 = (dt)^2 + r(t)^2 g_{S^m}, \quad r(t) = f(x(t)),$$

$$0 \le t \le t_0 = \int_0^{x_0} \sqrt{1 + f'(x)^2}dx.$$

**The Laplace-Beltrami operator on $M$ is**

$$\Delta_M = \frac{1}{r^m}\partial_t\left(r^m\partial_t\right) + \frac{\Delta_Y}{r^2},$$

**$\Delta_Y$ = Laplace-Beltrami op. on $S^m$.**
**We impose a suitable boundary condition on $t = 0, t = t_0$, and get the spectral data**

**eigenvalues + something related to eigenvectors**

## Inverse Spectral Problems

$$spectral\ data \Longrightarrow M$$

**Inverse procedure**

**(1) Compute $t_0$ from the eigenvalue asymptotics.**

**(2) Construct $r(t)$ on $[0, t_0]$.**

**(3) From the formulas**

$$r'(t) = f'(x(t))\frac{dt}{dx} = \frac{f'(x(t))}{\sqrt{1 + f'(x(t))^2}},$$

$$\frac{dt}{dx} = \sqrt{1 + f'(x)^2},$$

**we should have**

$$\frac{dx}{dt} = \sqrt{1 - r'(t)^2}.$$

**If we know $r(t)$, we can compute $x(t)$ and $x_0$.**

**(4) Use $r(t) = f(x(t))$ and the inverse function theorem to obtain $f(x)$.**

# Section 3. Liouville transform — global properties

**3.1 Rotationary symmetric manifold.**

**We slightly generalize the surface of revolution :**

$$M = [0,1] \times Y,$$

**$(Y, g_0)$ being a compact $m$-dim. Riemannian manifold (with or without boundary), equipped with the metric**

$$g = (dx)^2 + r^2(x) g_0,$$

**and the Laplace-Beltarmi operator**

$$\Delta_M = \frac{1}{r(x)^m} \partial_x \Big( r(x)^m \partial_x \Big) + \frac{\Delta_Y}{r^2(x)}.$$

**The boundary condition on $\partial M = \{0,1\} \times Y$ :**

$$\begin{cases} Dirichlet \quad f(0,y) = f(1,y) = 0, \\ Mixed \quad f(0,y) = f'(1,y) + bf(1,y) = 0, \\ Robin \quad f'(0,y) - af(0,y) = f'(1,y) + bf(1,y) = 0. \end{cases}$$

**$-\Delta_M$ has the discrete spectrum :**

$$0 \le E_1 \le E_2 \le \cdots .$$

**Letting $\rho = r^{m/2}$, $-\Delta_M$ is decomposed as**

$$-\Delta_M \simeq \oplus_{\nu=1}^{\infty}(-\Delta_\nu),$$

$$-\Delta_\nu = -\frac{1}{\rho^2}\partial_x(\rho^2\partial_x) + \frac{E_\nu}{r^2},$$

**acting on $L^2((0,1); r(x)^m dx)$.**

**The boundary condition is**

$$\begin{cases} Dirichlet \quad f(0) = f(1) = 0, \\ Mixed \quad f(0) = f'(1) + bf(1) = 0, \\ Robin \quad f'(0) - af(0) = f'(1) + bf(1) = 0. \end{cases}$$

**Let us introduce a parameter**

$$q_0 = \frac{\rho'(0)}{\rho(0)},$$

**and put**

$$\frac{\rho(x)}{\rho(x)} = q_0 + q(x).$$

**Then $r(x)$ is written as**

$$r(x) = r_0 e^{2Q(x)/m}, \quad Q(x) = \int_0^x (q_0 + q(t))dt.$$

We are going to determine the range of the spectral data mapping :

$$q \longrightarrow \{\mu_n(q), \kappa_n(q)\}_{n=1}^{\infty},$$

$\mu_n$ = eigenvalue, $\kappa_n$ = norming constant.

This seems to be easy, since this problem is reduced to the Schrödinger equation by the Liouville transform.

However,

Global properties of Liouville transform ?

- function spaces
- estimates for non-linear mapping

# Section 4. Function spaces

$$\mathcal{W}_1^0 = \{q \in L^2(0,1)\,;\, q' \in L^2(0,1), q(0) = q(1) = 0\},$$

$$\mathcal{W}_1^{0,odd} = \{q \in \mathcal{W}_1^0\,;\, q(x) = -q(1-x)\},$$

$$\mathcal{M}_1 = \{(h_n)_{n=1}^\infty \in \ell^2\,;\, \mu_1^0 + h_1 < \mu_2^0 + h_2 < \cdots\},$$

$$\mu_n^0 = (n\pi)^2,$$

$$\ell_1^2 = \Big\{(h_n)_{n=1}^\infty\,;\, \sum_{n=1}^\infty n^2|h_n|^2 < \infty\Big\}.$$

# Section 5. Main results 1 — Spectral data mapping

$$\begin{cases} -\Delta_\nu f = -\dfrac{1}{\rho^2}(\rho^2 f')' + \dfrac{E_\nu}{\rho^2}, \quad \rho = r^{m/2}, \\ f(0) = f(1) = 0. \end{cases}$$

## Eigenvalues

$$\mu_n = \mu_n^0 + c_0 + \widetilde{\mu}_n, \quad (\widetilde{\mu}_n) \in \ell_1^2,$$

$$c_0 = \int_0^1 \left((q_0 + q)^2 + \frac{E_\nu}{r^2}\right) dx,$$

## Norming constants

$$\kappa_n(q) = \log \left| \frac{\rho(1) f_n'(1,q)}{f_n'(0,q)} \right|.$$

## Theorem (Dirichlet B. C.)

**Fix $\nu \geq 1$, and assume either (i) or (ii),**

**(i) $q_0 = 0$,**

**(ii) $\nu = 1$ and $E_1 = 0$.**

**Then, the mapping**

$$\mathcal{W}_1^0 \ni q \to \left( (\widetilde{\mu}_n(q))_{n=1}^{\infty}, (\kappa_n(q))_{n=1}^{\infty} \right) \in \mathcal{M}_1 \times \ell_1^2$$

**is a real analytic isomorphism.**

**In particular, if the manifold $M$ is symmetric with respect to the plane $x = 1/2$,, the mapping**

$$\mathcal{W}_1^{0,odd} \ni q \to (\widetilde{\mu}_n(q))_{n=1}^{\infty}, \in \mathcal{M}_1$$

**is a real analytic isomorphism.**

The results for Mixed or Robin boundary conditions are similar.

# Section 6. Curvature mapping

## Minkowski problem

Existence of a convex surface with a prescribed Gaussian curvature.

In other words, given a strictly positive real function $F$ on $S^2$, one seeks a strictly convex compact surface $M$ whose Gaussian curvature at $x$ is equal to $F(n(x))$, where $n(x)$ is the unit normal at $x \in M$.

Recall that the profile of the rotationary symmetric manifold $M$ is represented as

$$r(x) = r_0 e^{2Q(x)}, \quad Q(x) = \int_0^x (q_0 + q(t))dt.$$

Then, the Gaussian curvature is written as

$$\begin{aligned} \mathcal{K} &= -\frac{r''}{r}, \quad \rho = r^{1/2}, \\ &= -2q' - 4(q_0 + q)^2. \end{aligned}$$

This is rewritten as

$$\mathcal{K} = -G(q) - \mathcal{K}_0 - 4q_0^2,$$

$$\mathcal{K}_0 = 4\int_0^1 (2q_0 q + q^2)dx,$$

$$G(q) = 2q' + 4(q_0 + q)^2 - \mathcal{K}_0.$$

**Theorem (Curvature mapping)**

**The mapping**

$$\mathcal{W}_1^0 \ni q \longrightarrow G(q) \in L^2(0,1)$$

**is a real analytic isomorphism. The constant $\mathcal{K}_0$ is uniquely determiend by $G(q)$.**

By this theorem, the mapping

$$\text{Gaussian curvature} \to \text{profile } r(x) \\ \to \text{eigenvalues} + \text{norming constants}$$

is well-defined.

Lets us consider the Strum-Liouville problem

$$-\frac{1}{\rho^2}(\rho^2 f')' + \frac{E_\nu}{r^2} f = \lambda f,$$

$$f'(0) - af(0) = f'(1) + bf(1) = 0.$$

We put

$$\xi = G(q) \in L^2(0,1),$$

$$A = (a, b, q_0) \in \mathrm{R}^3.$$

Then the eigenvalues are

$$\mu_n(\xi, A) = (n\pi)^2 + 2(a+b) + c_0 + \widetilde{\mu}_n(\xi, A),$$

$$c_0 = \int_0^1 \Big(q^2 + \frac{E_\nu}{r^2}\Big) dx,$$

$$(\widetilde{\mu}_n(\xi, A))_{n=1}^{\infty} \in \ell^2.$$

Norming constants are

$$\phi_n(\xi, A) = \log\left|\frac{\rho(1) f_n(1, \xi, A)}{f_n(0, \xi, A)}\right|,$$

$f_n$ being the $n$-th eigenfunction.

## Theorem (Spectral data mapping)

**Let $A = (a, b, q_0) \in \mathrm{R}^3$, $\nu \geq 1$ be fixed. Assume either (i) or (ii) :**

**(i) $q_0 = 0$,**

**(ii) $\nu = 1$ and $E_1 = 0$.**

**Then the mapping**

$$L^2(0,1) \ni \xi \longrightarrow \Big( (\widetilde{\mu}_n(\xi, A))_{n=1}^{\infty}, (\phi_n(\xi, A))_{n=1}^{\infty} \Big) \in \mathcal{M}_1 \times \ell_1^2$$

**is a real analytic isomorphism.**