Schrödinger flow into almost Hermitian manifolds

Hiroyuki CHIHARA

Department of Mathematics and Computer Science
Kagoshima University
hiroyuki.chihara@gmail.com

Geometric and Singular Analysis
March 24-28, 2014
University of Potsdam

Contents

0. Schrödinger evolution equation 1/2

Consider the IVP of the form

$$U_t - \sqrt{-1}\Delta_{\mathbb{R}^m} U = 0 \quad \text{in} \quad \mathbb{R} \times X, \tag{1}$$

$$U(0,x) = U_0(x) \quad \text{in} \quad X, \tag{2}$$

where $U(t,x)$ is $\mathbb{C}$-valued, and $X = \mathbb{R}^m$ or $X = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. The IVP (1)-(2) is well-posed in both directions in the past and the future, and the solution has the following property.

- If $X = \mathbb{R}^m$, then the solution U gains extra smoothness of order $1/2$ locally in space-time

$$\|(1+|x|)^{-1/2-\varepsilon}|D_x|^{1/2}U\|_{L^2(\mathbb{R}^{1+m})} \leqslant C\|U_0\|_{L^2(\mathbb{R}^m)}.$$

- If $X = \mathbb{T}^m$, then (1) has never smoothing effect.

Recall the classical orbit $(x + 2t\xi, \xi)$ in the phase space for $x \in X$ and $\xi \in \mathbb{R}^m \setminus \{0\}$.

0. Schrödinger evolution equation 2/2

Consider the IVP of the form

$$U_t - \sqrt{-1}\Delta_{\mathbb{R}^m} U + \vec{b}(x)\cdot\nabla_x U = 0 \quad \text{in} \quad \mathbb{R}\times X, \qquad (3)$$

$$U(0,x) = U_0(x) \quad \text{in} \quad X, \qquad (4)$$

where $\vec{b}(x) = (b_1(x),\dots,b_m(x)) \in \{C_b^\infty(X)\}^m$ and $U_0(x) \in L^2(X)$ are given functions. $\operatorname{Im}\vec{b}(x)$ is an obstuction to the well-posedness of (3)-(4). If this real vector firld has a scalar potential $\phi(x)$, that is,

$$\operatorname{Im} b_j(x) = \frac{\partial\phi}{\partial x_j}(x), \quad j = 1,\dots,m, \qquad (5)$$

then $V = e^{\phi}U$ solves

$$V_t - \sqrt{-1}\Delta_{\mathbb{R}^m} V + \{\operatorname{Re}\vec{b}(x)\}\cdot\nabla_x V + c(x)V = 0,$$

where $c(x)$ is a smooth function of $\vec{b}(x)$. The well-posedness for the IVP for this is obvious. In particular, in case of $X = \mathbb{T}^m$, (3)-(4) is L^2-well-posed if and only if (5) holds.

1. Schrödinger map equation

Consider the IVP of the form

$$u_t = J(u)\tau(u) \quad \text{in} \quad \mathbb{R} \times M, \tag{6}$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad M. \tag{7}$$

Notation

- $u : \mathbb{R} \times M \to N$.
- (M,g) is an m-dimensional closed Riemannian manifold.
- (N,J,h) is a $2n$-dimensional compact almost Hermitian manifold.
- $\tau(u)$ is the tension field of $u(t,\cdot) : M \to N$, i.e., roughly speaking,

$$\tau(u)(t,x) = \text{the projection of } \Delta_g u(t,x) \text{ on } T_{u(t,x)}N.$$

Remarks

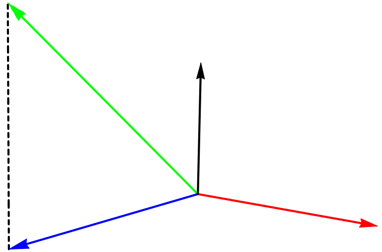
- A solution to (6) is said to be a Schrödinger map.
- $u \in C^\infty(M;N)$ is a steady solutions to (6) if and only if u is a harmonic map of M to N, i.e., $\tau(u) = 0$.

$$\vec{u}_t = \vec{u} \times \Delta_{\mathbb{R}^m} \vec{u}, \tag{8}$$

$$\vec{u} = (u_1, u_2, u_3) : \mathbb{R} \times \mathbb{R}^m \ni (t, x) \mapsto \vec{u}(t, x) \in \mathbb{S}^2.$$

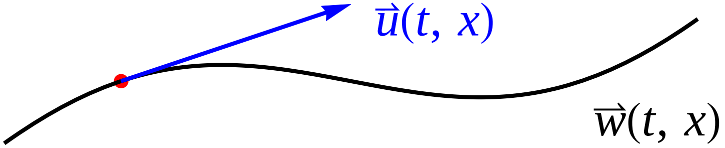
We can see $\vec{u} \times \Delta_{\mathbb{R}^m} \vec{u}$ as $J(\vec{u})\tau(\vec{u})$ since

- $\vec{u}$ is not only a point on $\mathbb{S}^2$, but also a unit normal vector at $\vec{u}$.
- $\vec{u} \times \Delta_{\mathbb{R}^m} \vec{u} = \vec{u} \times (\Delta_{\mathbb{R}^m} \vec{u})^T$,
  where $(\Delta_{\mathbb{R}^m} \vec{u})^T \in T_{\vec{u}(t,x)} \mathbb{S}^2$ is the tangent component of $\Delta_{\mathbb{R}^m} \vec{u}$.
- $\{\vec{u}, (\Delta_{\mathbb{R}^m} \vec{u})^T, \vec{u} \times (\Delta_{\mathbb{R}^m} \vec{u})^T\}$ is a right-handed orthogonal system in $\mathbb{R}^3$ if $(\Delta_{\mathbb{R}^m} \vec{u})^T \neq \vec{0}$:

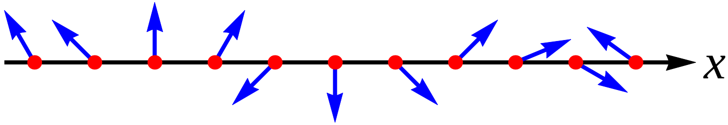


The PDE (8) models the following.

- the motion of vortex filament (Da Rios, 1906), where $m = 1$, $x \in \mathbb{R}$ is the arc-length, $\vec{u} = \vec{w}_x$, $\vec{w}(t, \cdot)$ is a curve of a vortex filament at the time t.



- the continuum limit of the Heisenberg spin chain system of ferromagnetism.



2. A model equation in classical mechanics 3/3

If $\vec{u}(t,x)$ never takes values in a neighborhood of the North Pole $(0,0,1) \in \mathbb{S}^2$, then (8) becomes a semilinear Schrödinger evolution equation of the form

$$v_t - \sqrt{-1}\Delta_{\mathbb{R}^m} v = \frac{2\sqrt{-1}\bar{v}}{1+|v|^2} \sum_{i=1}^{m} \left(\frac{\partial v}{\partial x^i}\right)^2 \tag{9}$$

via the stereographic projection

$$\mathbb{S}^2 \ni \vec{u} = (u_1, u_2, u_3) \mapsto v = \frac{u_1 + \sqrt{-1}u_2}{1-u_3} \in \mathbb{C}.$$

All the solutions to (9) are not interesting in classical mechanics. In particular, small solutions to (9) correspond to Schrödinger maps which are very close to constant maps.

3. Geometric analysis of Schrödinger maps

- AIM the structure of the Schrödinger map equation?
- Geometric Reduction
  - Chang-Shatah-Uhlenbeck (CPAM, 2000)
    $M = \mathbb{R}$, (N, J, h) is a compact Riemann surface, and
    $\exists u_\infty \in N$ s.t. $u(t,x) \to u_\infty$ as $x \to \infty$ for $\forall t \in \mathbb{R}$.
    $\Rightarrow \exists$ a moving frame along the curve $u(t,\cdot)$.
  - Nahmod-Shatah-Vega-Zeng (IMRN, 2008)
    $M = \mathbb{R}^m$, $\nabla^N J = 0 \Rightarrow \exists$ a moving frame along the map $u(t,\cdot)$.
- Analysis of the IVP
  - Sulem-Sulem-Bardos (CMP, 1985)
    Short-time existence and time-global weak solutions for (8).
  - Koiso (Osaka J. Math., 1997) $M = \mathbb{S}^1$ and $\nabla^N J = 0$.
    1. Reformulation of (8) as (6).
    2. Short-time existence.
    3. $\nabla^N R = 0 \Rightarrow$ Time-global existence.
- All the preceeding works assume the Kähler condition $\nabla^N J = 0$, which guarantees the classical energy estimates for (6).

Theorem 1

Let $k \in \mathbb{N}$ satisfying $2k > m/2 + 5$, and let $k_0 = \min k$.
Then, $\forall u_0 \in H^{2k}(M; TN)$, $\exists T = T(\|u\|_{H^{2k_0}}) > 0$, such that (6)-(7) has a unique solution $u \in C([-T, T]; H^{2k}(M; TN))$.

Remarks

- Set $G = \det[g_{ij}]$ and $\nabla_i = \nabla^N_{du(\partial/\partial x^i)}$. Then,
  $\tilde{\Delta}_g = \dfrac{1}{\sqrt{G}} \displaystyle\sum_{i,j=1}^{m} \nabla_i g^{ij} \sqrt{G} \nabla_j$ is globally well-defined.
- $\|u\|^2_{H^{2k}(M;TN)} \sim \displaystyle\sum_{l=1}^{k} \int_M h(\tilde{\Delta}^l_g u, \tilde{\Delta}^l_g u) d\mu_g, \quad \tilde{\Delta}^l_g u = \tilde{\Delta}^{l-1}_g \tau(u).$
  This is used for avoiding some kinds of a loss of one derivative.
- $u \in H^{2k_0}(M; TN) \Rightarrow \nabla_i u \in C^4(M; TN)$.
  Our calculus requires this smoothness of $\nabla_i u$.

4. Short-time existence without $\nabla^N J = 0$ 2/5

We show the idea of proof. Set $v = \tilde{\Delta}_g^{k-1}\tau(u)$. Then,

$$\left\{\nabla_t - \frac{1}{\sqrt{G}}\sum_{i,j=1}^{m}\nabla_i g^{ij}\sqrt{G}J(u)\nabla_j - P\right\}v = \text{harmless terms}, \quad (10)$$

$P = (2k-1)\sum_{i,j=1}^{m} g^{ij}\{\nabla_i J(u)\}\nabla_j$ is an anti-symmetric first-order term, and an obstruction to establishing short-time existence theorem.

Here we recall the Cauchy-Riemann system

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Note that (10) has no smoothing effect since the sauce manifold (M, g) is compact. Fortunately, however, we can make eliminate P from (10) by using some gauge transform on $u^{-1}TN$.

4. Short-time existence without $\nabla^N J = 0$ 3/5

Our Idea: Consider

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left\{ \sqrt{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial^2}{\partial x^2} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \right\} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The simbol/$\sqrt{-1}$ of the right hand side is

$$\begin{bmatrix} \xi^2 & \xi \\ -\xi & -\xi^2 \end{bmatrix} = \xi^2 \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\xi}\right) \right\} \quad |\xi| \to \infty.$$

This can be diagonazed for large ξ.

4. Short-time existence without $\nabla^N J = 0$ 4/5

Here we introduce a PsDO $\Lambda = 1 - \tilde{\Lambda}$ acting on $\Gamma(u^{-1}TN)$ as follows. $\tilde{\Lambda}$ is of order -1. Rougly speaking,

$$\tilde{\Lambda} = \frac{PJ(u)(1-\Delta_g)^{-1}}{2} \simeq -\frac{J(u)P(1-\Delta_g)^{-1}}{2}$$

locally. Fortunately, this expression is invariant under the change of variables of M and N, and makes sense globally. However, here is a problem regarding PsDOs:

- $\tilde{\Lambda}$ is a PsDO on manifolds.
- $\tilde{\Lambda}$ is a PsDO with limited smoothness.

The requiremments of $\tilde{\Lambda}$ for these are contrary from a point of view of so-called the type (ρ, δ) of PsDOs. For this reason, we construct $\tilde{\Lambda}$ by using not only the partition of unity on M but also that on N. $\tilde{\Lambda}$ is uniformly poperly supported, and $\tilde{\Lambda}$ behaves like a PsDO acting on $\mathbb{R}^{2n}$-valued functions on $\mathbb{R}^m$.

Set

$$\tilde{\Lambda} = \sum_{\alpha}\sum_{\beta} \phi_\alpha(x)\psi_\beta(u)\tilde{\Lambda}_{\alpha,\beta}\Phi_\alpha(x)\Psi_\beta(u),$$

where

$$\phi_\alpha, \Phi_\alpha \in C_0^\infty(M), \quad \sum_{\alpha}\phi_\alpha = 1, \quad \Phi_\alpha = 1 \text{ in } \operatorname{supp}[\phi_\alpha],$$

$$\psi_\beta, \Psi_\beta \in C_0^\infty(N), \quad \sum_{\beta}\psi_\beta = 1, \quad \Psi_\beta = 1 \text{ in } \operatorname{supp}[\psi_\beta],$$

and $\tilde{\Lambda}_{\alpha,\beta}$ is a local expression of $\tilde{\Lambda}$. Then, Λ works well and

$$\left\{\nabla_t - \frac{1}{\sqrt{G}}\sum_{i,j=1}^{m}\nabla_i g^{ij}\sqrt{G}J(u)\nabla_j\right\}\Lambda v = \text{harmless terms.}$$

5. The Kähler condition and PDEs

- Kähler Case: $\nabla^N J = 0$.
  The square of the L^2-norm $\int_M h(v,v)\,d\mu_g$ for $v \in \Gamma(u^{-1}TN)$ corresponds to $\int_{\mathbb{T}^m} e^{2\phi}|U|^2 dx$ for $U \in L^2(\mathbb{T}^m)$.
  No obstruction for the well-posedness.
- Non-Kähler Case: $\nabla^N J \neq 0$.
  Theorem 1 says that the Schrödinger map equation admits some other kinds of (seeming) obstruction since the equation is a system.
  In other words, $\nabla^N J \neq 0$ is a weak obstruction for solving the IVP.