Fourth-order linear dispersive systems and dispersive flows into Riemann surfaces

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Contents

1. IVP for a linear dispersive system

$$L\vec{u} \equiv \left\{ I\frac{\partial}{\partial t} + J\frac{\partial^4}{\partial x^4} + B(x)\frac{\partial^2}{\partial x^2} + C(x)\frac{\partial}{\partial x} \right\} \vec{u} = \vec{f}(t,x), \quad (1)$$

$$\vec{u}(0,x) = \vec{u}_0(x), \quad (2)$$

- where
  - $\mathbb{R} \times \mathbb{T} \ni (t,x) \mapsto \vec{u}(t,x) \in \mathbb{R}^2$ unknown,
  - $\vec{f}(t,x)$, $\vec{u}_0(x)$ given, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,
  - I, J, $B(x)$ and $C(x)$ are 2×2 real matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B(x), C(x) \in C^\infty(\mathbb{T}; M_2(\mathbb{R})).$$

- $B(x) + {}^tB(x) = 0, \quad C(x) - {}^tC(x) = 0$

  $\Rightarrow$ Energy estimates work for (1).

2. Dispersive flows into Riemann surfaces

$$
\begin{aligned}
u_t &= a\tilde{J}_u \nabla_x^3 u_x + \{1 + b g_u(u_x, u_x)\}\tilde{J}_u \nabla_x u_x \\
&\quad + c g_u(\nabla_x u_x, u_x)\tilde{J} u_x,
\end{aligned} \tag{3}
$$

- where
  - $\mathbb{R} \times \mathbb{T} \ni (t, x) \mapsto u(t, x) \in N$ unknown,
  - $(N, \tilde{J}, g)$ a compact Riemann surfaces,
  - $a, b, c \in \mathbb{R}$, $a \neq 0$ constants,
  - ∇_t, ∇_x covariant derivatives along u.
- (3) models the motion of vortex filaments and spin-chain systems.
- Onodera: The sectional curvature is constant.

  $\Rightarrow$ short-time existence (uniqueness ?).
- $a = b = c = 0 \quad \Rightarrow \quad$ (3) becomes $u_t = \tilde{J}\nabla_x u_x$.
- C (2013): short-time existence of the Schrödinger flow from closed Riemannian manifolds into compact almost Hermitian manifolds.

3. Well-posedness of the IVP

Theorem

The following conditions are mutually equivalent.

(I) *For any* $\vec{u}_0 \in L^2(\mathbb{T};\mathbb{R}^2)$ *and for any* $\vec{f} \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{T};\mathbb{R}^2))$, *the IVP* (1)-(2) *has a unique solution* $\vec{u} \in C(\mathbb{R}; L^2(\mathbb{T};\mathbb{R}^2))$.

(II) $\displaystyle\int_0^{2\pi} \mathrm{tr}(B(x))dx = \int_0^{2\pi} \mathrm{tr}(JC(x))dx = 0.$

- $\mathrm{tr}(B(x)) = b_{11}(x) + b_{22}(x)$, $\mathrm{tr}(JC(x)) = c_{12}(x) - c_{21}(x)$.
- (I) implies the continuity of $(\vec{u}_0, \vec{f}) \mapsto \vec{u}$.
- Proof

  Consider more general systems. By using PsDOs, we can diagonalize (1) essentially. The proof is reduced to Mizuhara's results on single equations (2006).

4. Direct proof of the sufficiency 1/2

- Split a 2×2 matrix A into three parts:

$$A = \underbrace{\frac{\operatorname{tr}(A)}{2} I + \frac{1}{2}\{JA - AJ\}J}_{(A+{}^tA)/2} \underbrace{- \frac{\operatorname{tr}(JA)}{2} J}_{(A-{}^tA)/2}.$$

- Eliminate $\frac{\operatorname{tr}(B(x))}{2} I$, $\frac{1}{2}\{JB(x) - B(x)J\}J$, $-\frac{\operatorname{tr}(JC(x))}{2} J$.
- Let $r \gg 1$. Pick up $\varphi_r(\xi) \in C^\infty(\mathbb{R};\mathbb{R})$ so that $\varphi_r(-\xi) = \varphi_r(\xi)$,

$$\varphi_r(\xi) = 1 \quad (|\xi| \geqslant r+1), \quad \varphi_r(\xi) = 0 \quad (|\xi| \leqslant r).$$

- Set $p_\ell(\xi) = \varphi_r(\xi)/(\sqrt{-1}\xi)^\ell$, $\ell = 1, 2, 3, \dots$.
- $\operatorname{Im} v = 0 \quad \Rightarrow \quad \operatorname{Im} p_\ell(D_x) v = 0$, where $D_x = -\sqrt{-1}\partial/\partial x$.
- Suppose (II). Define smooth 2π-periodic functions by

$$\Psi_1(x) = \int_0^x \operatorname{tr}(B(y))dy, \quad \Psi_3(x) = \int_0^x \operatorname{tr}(JC(y))dy + \cdots.$$

4. Direct proof of the sufficiency 2/2

- Define a PsDO Λ by

$$\Lambda = (I - \tilde{\Lambda}_3)(I - \tilde{\Lambda}_2)(I - \tilde{\Lambda}_1),$$
$$\tilde{\Lambda}_1 = \frac{1}{8}\Psi_1(x) J p_1(D_x),$$
$$\tilde{\Lambda}_2 = -\frac{1}{2}\left\{ B(x) - \frac{\operatorname{tr}(B(x))}{2} I \right\} J p_2(D_x),$$
$$\tilde{\Lambda}_3 = -\frac{1}{4}\Psi_3(x) I p_2(D_x).$$

- Λ is invertible on $L^2(\mathbb{T};\mathbb{R}^2)$ for $r \gg 1$, and

$$\Lambda L \Lambda^{-1} = I\frac{\partial}{\partial t} + J\frac{\partial^4}{\partial x^4} - \frac{\partial}{\partial x}\frac{\operatorname{tr}(JB(x))}{2} J \frac{\partial}{\partial x} + C_1(x)\frac{\partial}{\partial x} + \cdots,$$

where $C_1(x)$ is a symmetric matrix.

5. Moving frames along closed curves

- Let $u \in C^\infty(\mathbb{R} \times \mathbb{T}; N)$ be a solution to (3).
  Suppose that $u_x(t, 0) \neq 0$ for all $t \in \mathbb{R}$. Set

$$e_0(t) = \frac{u_x(t,0)}{\sqrt{g_{u(t,0)}(u_x, u_x)}}, \quad \gamma(t) = \{u(t,x) \mid x \in \mathbb{T}\}.$$

- Let $e(t, x)$ be a solution to $\nabla_x e = 0$, $e(t, 0) = e_0(t)$.
- $\{e, \tilde{J}e\}$ is a moving frame along $\gamma(t)$.
- $\nabla_x \tilde{J}e = 0$, $\tilde{J}e(t, 0) = \tilde{J}e_0(t)$ since $\nabla^N \tilde{J} = 0$.
- Note that $e(t, 0) \neq e(t, 2\pi)$ in general.
- Let $2\pi\theta(t)$ be the holonomy angle of $\gamma(t)$.
  If $\gamma(t)$ is a boundary of contractive domain $\Omega(t)$, then

$$\theta(t) = \frac{1}{2\pi} \int_{\Omega(t)} K(u) du^1 \wedge du^2,$$

  where $K(u)$ is the sectional curvature at $u \in N$.

6. Geometric reduction of dispersive flows 1/3

Theorem

If $K(u)$ is constant, then (3) has nice structure so that the IVP is solvable.

- History of geometric reduction
  - Chang-Shatah-Uhlenbeck (2000): the Schrödinger map equation.
  - Onodera (2008): third and fourth-order equations.
  - The case of $x \in \mathbb{R}$ with a base point $u_\star = \lim_{x\to\infty} u(t,x)$.
  - A $\mathbb{C}$-valued equation is obtained from that for u_x.
    It is hard to distinguish the unknown and the coefficients.
    It is hard to know that $\sqrt{-1}$ corresponds to $\tilde{J}$ or not.
- We shall obtain a 2×2 system from the equation for a higher order derivative $\nabla_x^\ell u_x$, $\ell = 4, 5, 6, \dots$.

6. Geometric reduction of dispersive flows 2/3

- Set $\nabla_x^{\ell} u_x = Ve + W\tilde{J}e$ for $\ell = 4, 5, 6, \dots$.
- Let R be the Riemann curvature tensor of $(N, \tilde{J}, g)$.
- Using the following properties

$$R(\tilde{J}e, e)e = K(u)\tilde{J}e, \quad \tilde{J}R(\tilde{J}e, e)e = R(\tilde{J}e, e)\tilde{J}e, \dots$$

we obtain

$$\left\{ I\frac{\partial}{\partial t} - aJ\frac{\partial^4}{\partial x^4} + \hat{B}(t,x)\frac{\partial^2}{\partial x^2} + \hat{C}(t,x)\frac{\partial}{\partial x} \right\} \begin{bmatrix} V \\ W \end{bmatrix} = \cdots,$$

- $\operatorname{tr}\left(\hat{B}(t,x)\right) \equiv 0$,

$$\int_0^{2\pi} \operatorname{tr}\left(J\hat{C}(t,x)\right)dx = -\frac{a}{2}\int_0^{2\pi} g_u(u_x, u_x)\frac{\partial}{\partial x}K(u)dx.$$

6. Geometric reduction of dispersive flows 3/3

- Correction

$$\vec{z} = \begin{bmatrix} \cos\bigl(\theta(t)x\bigr) & -\sin\bigl(\theta(t)x\bigr) \\ \sin\bigl(\theta(t)x\bigr) & \cos\bigl(\theta(t)x\bigr) \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}.$$

- $\vec{z} \in C^\infty(\mathbb{R} \times \mathbb{T}; \mathbb{R}^2)$ and $\vec{z}$ solves

$$\Bigl\{ I \frac{\partial}{\partial t} - aJ \frac{\partial^4}{\partial x^4} - a\theta(t) I \frac{\partial^3}{\partial x^3} + \hat{B}_1(t,x) \frac{\partial^2}{\partial x^2} + \hat{C}_1(t,x) \frac{\partial}{\partial x} \Bigr\} \vec{z} = \cdots ,$$

- $\operatorname{tr}\bigl(\hat{B}(t,x)\bigr) = \operatorname{tr}\bigl(\hat{B}_1(t,x)\bigr), \quad \operatorname{tr}\bigl(J\hat{C}(t,x)\bigr) = \operatorname{tr}\bigl(J\hat{C}_1(t,x)\bigr).$

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